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Large systematic deviations in visual parallelism

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Abstract. The visual environment is distorted with respect to the physical environment. Luneburg [1947, *Mathematical Analysis of Binocular Vision* (Princeton, NJ: Princeton University Press)] assumed that visual space could be described by a Riemannian space of constant curvature. Such a space is described by a metric which defines the distance between any two points. It is uncertain, however, whether such a metric description is valid. Two experiments are reported in which subjects were asked to set two bars parallel to each other in a horizontal plane. The backdrop consisted of wrinkled black plastic sheeting, and the floor and ceiling were hidden by means of a horizontal aperture restricting the visual field of the subject vertically to 10 deg. We found that large deviations (of up to 40°) occur and that the deviations are proportional to the separation angle: on average, the proportion is 30%. These deviations occur for 30°, 60°, 120°, and 150° reference orientations, but not for 0° and 90° reference orientations; there the deviation is approximately 0° for most subjects. A Riemannian space of constant curvature, therefore, cannot be an adequate description. If it were, then the deviation between the orientation of the test and the reference bar would be independent of the reference orientation. Furthermore, we found that the results are independent of the distance of the bars from the subject, which suggests either that visual space has a zero mean curvature, or that the parallelity task is essentially a monocular task. The fact that the deviations vanish for a 0° and 90° orientation is reminiscent of the oblique effect reported in the literature. However, the 'oblique effect' reported here takes place in a horizontal plane at eye height, not in a frontoparallel plane.

1 Introduction

In daily life, we identify the visually perceived world with the physical world. The reason for this is straightforward: visually directed behaviour, such as knowing whether an object is in grasping range, does not pose any serious problems. Scientifically, this identification is not so trivial unless the perceived spatial relations, such as the perceived distance between objects, the perceived straightness of a line, and so on, resemble the physical spatial relations. However, throughout the literature it has been shown that the perceived geometrical relations deviate significantly from the physical relations. For example, Helmholtz (1867/1962) showed that wires which are arranged in an apparent frontoparallel plane do not lie in a physically frontoparallel plane; Hillebrand (1902) found that two lines in depth which appear equidistant are not physically equidistant; and Blumenfeld (1913) found that such equidistant lines do not appear straight and parallel. Evidently, the visually perceived space is distorted with respect to physical space. The correspondence is probably systematic, for otherwise it is hard to understand why visual perception does not pose problems in daily life. However, it is doubtful whether there is a one-to-one correspondence. Therefore, we still have to determine which geometrical relations exist between visual percepts and find out to what extent visual space corresponds to physical space.

In Euclidean geometry of the plane, the sum of the interior angles of a triangle equals π radians and parallel lines are equidistant. It was found, however, that these relations do not hold in visual space (Blumenfeld 1913; Foley 1980, 1991; Indow 1991) and that visual space therefore cannot be described by a Euclidean geometry (Luneburg 1947;

Blank 1978; Indow 1997). Luneburg (1947) introduced a Riemannian space of constant curvature as a description for the visually perceived space or, in short, visual space.

In such a space, parallel lines are not necessarily equidistant, and the sum of the interior angles of a triangle is not necessarily π radians but may depend on the area of the triangle. The geometrical relations are determined by the curvature (Stoker 1969). There are only three types of qualitatively different constantly curved two-dimensional Riemannian spaces: planar, spherical, and saddle-shaped surfaces. In the case of a spherical surface, straight lines are replaced by great circles. As a result, equidistant lines are no longer straight and parallel: the circles of constant latitude are all equidistant from the equator but none of them is a great circle (except the equator itself). Also, the sum of the interior angles of a triangle exceeds π. Take, for example, the triangle with the 0° and 90° meridian and the equator as its sides. In this case the sum of the interior angles equals $\frac{3}{2}\pi$. The difference of $\frac{1}{2}\pi$ radians equals the area of the triangle $\frac{1}{2}\pi r^2$ times the curvature of the surface $1/r^2$.

Qualitatively, Luneburg's model explained why parallel alleys differed from distance alleys. The remaining problem was to determine the curvature. This was attempted extensively in the literature by means of experiments in a dark room in which faint luminous light points were used as stimuli (Zajaczkowska 1956a, 1956b; Blank 1958, 1961; Hardy et al 1951; Indow et al 1962a, 1962b; Indow and Watanabe 1984; Indow 1997). The results, however, are inconclusive because of the large differences between subjects and between the experiments (Indow 1991). Moreover, some of the assumptions Luneburg made turned out to be invalid. For example, lights arranged on a Vieth–Müller circle are not seen, as was assumed, to be lying on an equidistant circle with the observer at its centre (Foley 1966; Heller 1997b). The constant-curvature condition was generally not satisfied either (Foley 1963, 1972; Higashiyama 1981, 1984; Koenderink and van Doorn 1998). Improvements to Luneburg's model were proposed by Blank (1978), Foley (1980, 1991), and Heller (1997a), but Lukas (1983) found that Luneburg's model provided a better description for frontoparallel horopters than either Blank's or Foley's model. On the other hand, experiments conducted outdoors with redundant monocular and binocular cues revealed that the relation between the perceived and physical distance was incompatible with Luneburg's model (Gilinsky 1951; Battro et al 1976, 1978; Wagner 1985). Thus, in any case, a distinction has to be made between a context-free and a context-rich environment.

A more fundamental assumption in Luneburg's model is that visual space is metric. A metric is a function which determines the length of a vector and the angle included by two vectors (Stoker 1969). In Euclidean geometry this is simply the inner product. All important geometrical relations can be derived from the metric. For instance, the distance between two points is determined uniquely by the metric. However, it is not clear whether such a definition is possible in visual space. For example, if the visual system used only occlusion of objects as a cue to distance, then it would be possible to determine for convex objects which one is closer when one object occludes the other, but it is impossible to determine from three occluding objects which ones are separated by the largest distance, the first two or the last two. In such a visual space, there is a rank ordering in depth, but it is impossible to talk about the distance between objects or about a metric.

If we wish to investigate whether visual space is metric, the task must not imply the existence of a metric, as would be the case when distances are estimated directly. For this purpose, we use a task in which the subject is asked to adjust the orientation of a test bar such that it appears to have the same orientation as the reference bar, both bars being arranged in a horizontal plane. In other words, the bars have to be set parallel in the sense of Levi-Civita (Stoker 1969), which is a valid definition of parallelity in any curved space. In principle, this task can be carried out by comparing the retinal image size of the bars; therefore the task does not require the existence of a metric, but, in order to make veridical judgments, distance estimates need to be made.

In this paper, we measured which bar orientations were judged parallel when the bars were placed in a horizontal plane at eye height. Previously, it had been shown that a pointing task, where a pointer was directed to a target, was very useful for obtaining properties of visual space (Koenderink and van Doorn 1998; Cuijpers et al, in press). The advantage of the present task is that parallelism lies at the heart of Riemannian geometries. If visual space is Riemannian, two bars which appear parallel should also appear parallel when rotated over the same visual angle. If, in addition, visual space is constantly curved, the corresponding physical angles are also the same. As a consequence, the deviation between the test and the reference bar should be independent of the orientation of the reference bar. This was tested in experiment 1, where the orientation of the reference bar and the separation between the test and reference bar was varied. This manner of testing the constant-curvature condition differs from previously reported methods (Eschenburg 1980). Another property of a curved Riemannian space is that, if a vector is parallel-transported along a closed loop, an angular defect will arise. Thus, if visual space is curved, subsequently setting bars parallel to each other along a closed loop will result in different apparent orientations before and after circumnavigating the loop. Consequently, the corresponding physical orientation will also differ. In fact, the angular defect will be positive, zero, or negative when the curvature of the visual space is positive, zero, or negative. In order to determine the angular defect, we performed experiment 2 in which the distance of the test bar relative to the reference bar was varied.

2 General methods

2.1 *Subjects*

The subjects were eight naïve undergraduate students, all in their early twenties. Four subjects participated in each experiment. All subjects had normal or corrected-to-normal visual acuity better than visus 1 and stereo vision with better than 60 s of arc acuity for a standard TNO test (Walraven 1975). The measurements were conducted binocularly. The subjects received no feedback about their performance.

2.2 *Experimental setup*

The measurements took place in a 6 m by 6 m room with blinded windows and normal room-lighting conditions. The walls were covered with black plastic sheet material arranged so that the corners of the room were hidden and the background looked similar in all directions. The plastic was wrinkled in order to create a heavy 'random' relief: the extent of the ridges and cavities was of the order of 10 cm (see figure 1).

The subject was seated on a chair of adjustable height placed in a small cabin. The cabin consisted of three wooden side walls and a small roof. The back was open. Once seated inside the cabin, the subject could look through a horizontal opening (with a height of about 10 cm) between the sides and the roof. As a result, both the floor and the ceiling of the room were invisible. The visual field of the subjects extended about 10 deg vertically (see figure 2) and 210 deg horizontally. The sides and the roof of the cabin were also covered with the black plastic sheet material.

The orientation of the subject was fixed by means of a chinrest that was mounted inside the cabin.

2.3 *Stimuli*

Four pairs of identical bars scaled in size were used as stimuli. Each bar was placed at a specific distance from the observer such that the apparent size was equal for all distances. The distances that were used were 1.47 m, 2.10 m, 3.00 m, and 4.31 m (see table 1). Each bar consisted of a rod with a pointed tip at each end (top angle 60°),

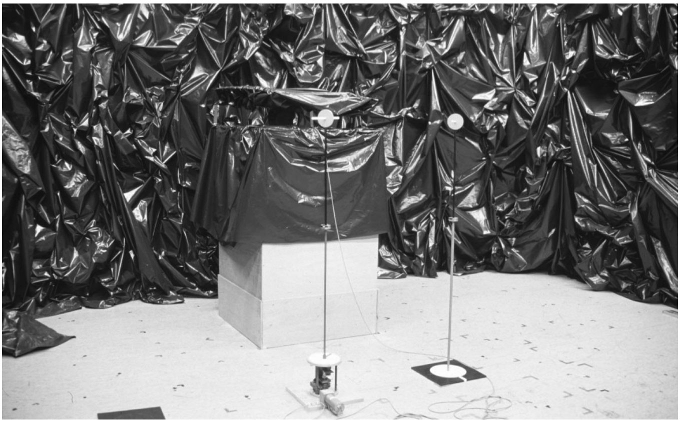

Figure 1. Picture of the experimentation room with the cabin. The walls are covered with black, wrinkled plastic. The subject is seated inside the cabin, the roof and sides of which prevent the subject from seeing the floor. In front of the cabin are the test bar (left) and the reference bar (right), which are mounted on thin vertical metal rods at a height of 1.38 m.

Figure 2. Picture from inside the cabin of the two bars that appear parallel to the subject. The visual field extended about 10 deg vertically and 210 deg horizontally.

Table 1. Dimensions of the bars for each distance from the subject. The size of each bar was scaled with the distance.

Distance/m	Rod		Disk	
	length/mm	diameter/mm	width/mm	diameter/mm
1.47	122	5	5	40
2.10	175	7	7	57
3.00	250	10	10	82
4.31	359	14	14	118

which protruded at right angles on each side of a circular disk. The rod was painted white and the disk yellow. The bars were mounted horizontally at eye height on thin vertical metal rods and could be rotated in the horizontal plane. The thin vertical metal rods were painted black. Their diameter was the same as the diameter of the rod of the bar. This construction could be placed on another vertical metal rod connected to a motor which was operated by the subject by remote control, or, alternatively, on a vertical metal rod the orientation of which was adjusted manually by the experimenter. The bar orientation could be read off from a scale that was invisible to the subject.

Markers were placed on the floor to position the bars. For experiment 1, the positions were selected from the intersections of three lines of constant radius (1.47 m, 2.10 m, and 4.31 m), centred on the subject, with five lines of constant angle ($\pm 30°$, $\pm 15°$, and $0°$), emanating from the subject (see figure 3a). In experiment 2, an additional distance of 3.00 m was used at an angle of 30°. The subjects were positioned such that the chinrest was directly above the origin and the median line was in the 0° direction. The front of the cabin was perpendicular to the median line.

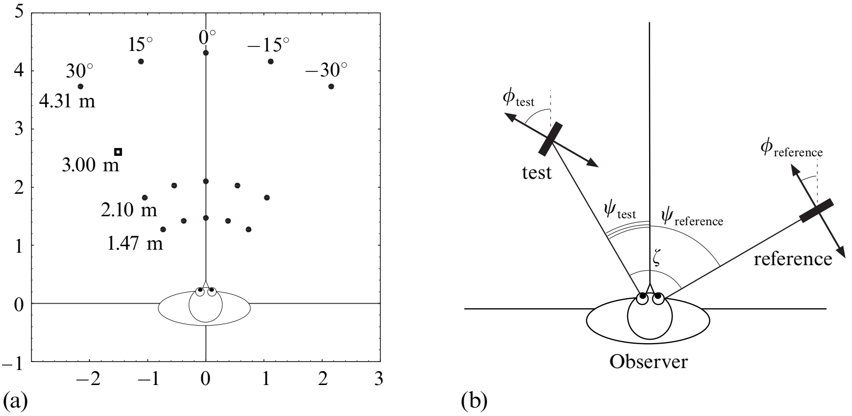

Figure 3. (a) Schematic diagram of the top view of the room. The bar positions that were used are indicated by the black dots. The square indicates the additional position used in experiment 2. The subject is seated with the chin directly above the origin. Both Cartesian and polar coordinates are indicated. (b) Diagram showing the definition of the orientation ϕ of both the test bar and the reference bar. The relation between the separation angle ζ and the polar angles ψ of the bars is also shown.

2.4 *Procedure*

The subjects were asked to adjust a test bar, operated by remote control, such that it appeared parallel to a reference bar. Many definitions of the term 'parallel' in common language are unsuitable for curved spaces. Therefore we used a drawing on a piece of paper to instruct the subjects, and we made clear that bars with the same physical orientation are physically parallel. With this definition of parallelity we ensured that the bars were set parallel in the sense of Levi-Civita. Before entering the room, the subjects were asked to cover their eyes; when seated on the chair, they were allowed to see again. Consequently, the subjects could observe the room from a prescribed vantage point only. In experiment 2, we also disoriented the subjects by rotating them about their axis when they entered the room while they were still blindfolded. The subjects had to keep their eyes closed while the test and reference bar were being positioned. Once the bars were in position, the subjects were asked to rotate the test bar an arbitrary amount with their eyes still closed. In the meantime the orientation of the reference bar was set by the experimenter. Upon a signal from the experimenter, the subjects opened their eyes and adjusted the orientation of the test bar until it appeared parallel to the reference. The subjects signalled when they were satisfied and closed their eyes again. After that, the orientation was noted and the following trial was set up.

The orientation of the bar is expressed by the angle ϕ between the line through the bar and the median line (see figure 3b). The angle ϕ varies from 0° to 180° because each bar points to either side. Instead of the actual orientation, we will be interested mainly in the difference $\Delta\phi$ in the orientation of the test bar and the reference bar, defined as $\Delta\phi = \phi_{\text{test}} - \phi_{\text{reference}}$ (mod 180°). The position of each bar is expressed in its polar coordinates, ie $(r_{\text{test}}, \psi_{\text{test}})$ and $(r_{\text{reference}}, \psi_{\text{reference}})$. In addition to the polar angles we will also use the separation angle defined by $\zeta = \psi_{\text{test}} - \psi_{\text{reference}}$ and the angle $\omega = \frac{1}{2}(\psi_{\text{test}} + \psi_{\text{reference}})$, which is the angle between the bisecting line and the median line (0°-line). The angle ω reflects the orientation in space of the stimulus positions with respect to the subject.

3 Experiment 1

In experiment 1 we investigated whether apparent parallelism departs from veridical and, if so, in what way. An important question here is whether the deviations we may find will depend on the orientation of the reference bar. This is important because models which assume that visual space can be described by a Riemannian metric, such as Luneburg's model (1947), predict that these deviations will be independent of the reference orientation.

3.1 *Method*

The test bar and the reference bar were always positioned at the same distance from the observer. The distances used were 1.47 m, 2.10 m, and 4.31 m. At each distance, the reference bar was placed at one of three polar angles (see figure 3a): 30°, 0°, or −30°. The test bar was placed at the same distance at one of the four remaining positions: 30°, 15°, 0°, −15°, or −30°, but without the reference position. For each configuration of reference bar and test bar positions, six different reference orientations were measured: 0°, 30°, 60°, 90°, 120°, and 150°. This results in a total of 216 different trials. All trials were repeated three times and measured for four subjects. The total measuring time was about 38 h.

The presentation order was as follows: for each stimulus configuration, the six different reference orientations were presented in sequence, but each time in a different random order. After these six trials a new position was selected at random for the test bar at the same distance. For this configuration again, six reference orientations were measured. This was repeated until all four positions of the test bar had been measured. Then a new reference position was chosen at random (at any distance) and the same procedure was repeated until all 216 trials had been performed. This was then repeated three times with a different randomisation for each repetition. The measurements were also in a different order for each subject. We used this presentation order instead of a completely randomised order, because in the latter case the measuring time would have been about twice as long.

3.2 *Results*

In figure 4 both graphical and numerical representations of the results are shown for subject AV for a distance of 1.47 m and a reference orientation of 30°. The results for the other subjects and distances are similar and will be discussed in more detail later on. In each figure, the reference bar is indicated by a thick line and the subject by a triangle. The average orientation of the test bar is shown graphically in the top row for four positions, and the deviations from veridical are indicated numerically in the bottom row. The length of the bars is exaggerated (3.4 times). The reference position ($\psi_{\text{reference}}$) is −30°, 0°, 30° in the left, middle, and right columns, respectively. It can be seen that, when the test bar is positioned next to the reference bar (a separation of $\zeta = \pm 15°$), there is a small difference between the orientation of the test and reference. On the other hand, when the test bar is positioned furthest away from the reference bar (a separation of $\zeta = \pm 60°$), there is a much larger difference. This is similar for all three reference positions (see figure 4). Therefore we plot the data as a function of the separation angle $\zeta = \psi_{\text{test}} - \psi_{\text{reference}}$.

In figure 5 the deviation $\Delta\phi$ from physically parallel is shown as a function of the separation angle ζ. The data are shown for subject AV and a stimulus distance of 1.47 m only. Each graph in figure 5 corresponds to a different reference orientation which is indicated in the top left corner. The data points correspond to single measurements

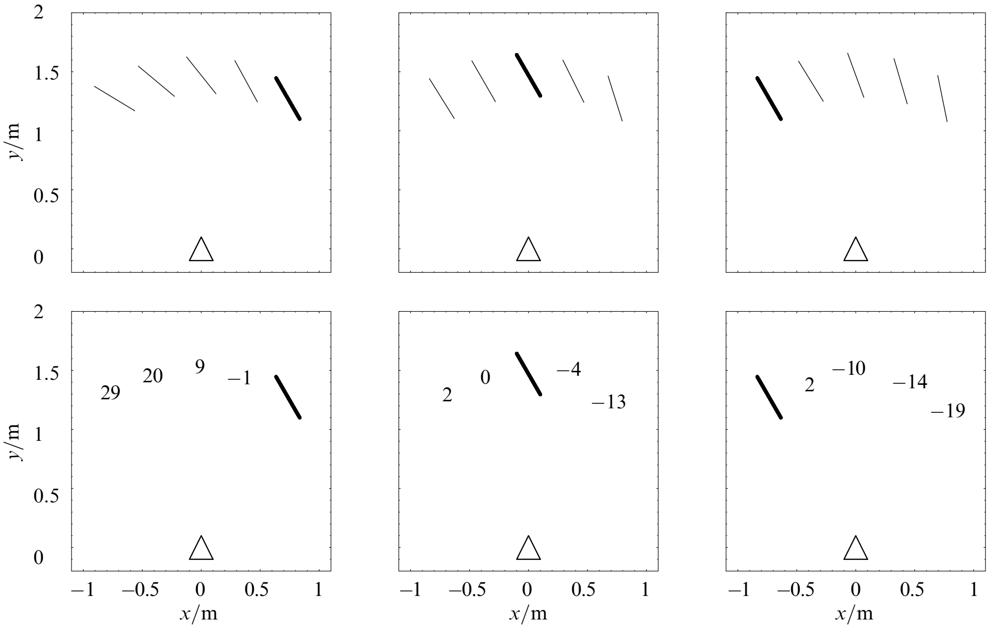

Figure 4. Graphical and numerical representation of the results for subject AV when the reference bar has an orientation of 30°. The distance of both bars from the subject is 1.47 m. In each figure of the top row the orientation is shown for four positions of the test bar (thin lines) and one position of the reference bar (thick line). In the bottom row the deviation $\Delta\phi$ is shown numerically for each position of the test bar. The position of the subject is indicated by a triangle. From left to right the reference position is $-30°$, $0°$, and $30°$, respectively.

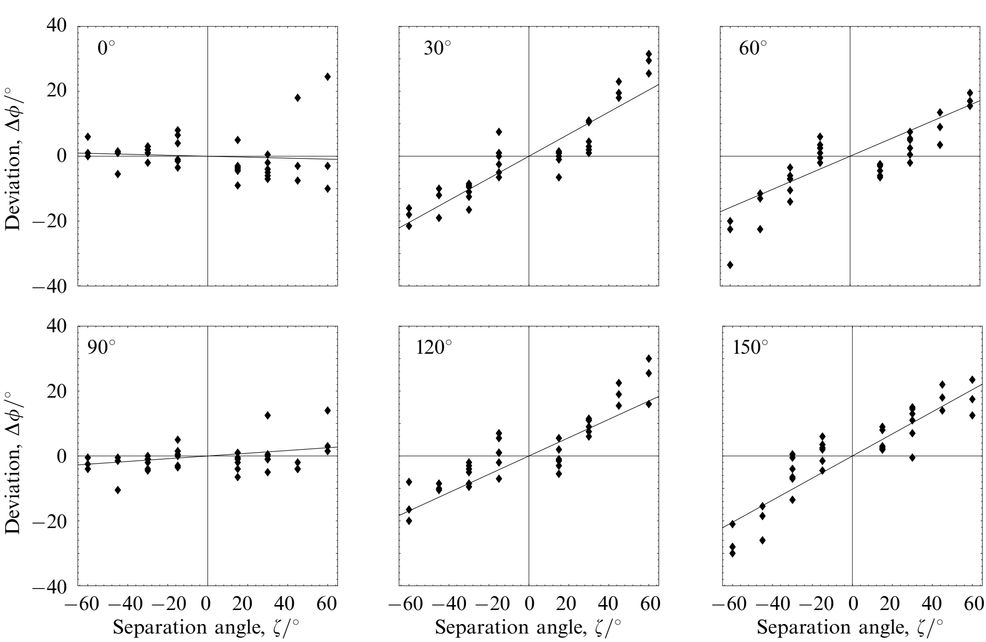

Figure 5. Deviation, $\Delta\phi$, between the orientation of the test bar and the reference bar as a function of the separation angle ζ. The data correspond to a stimulus distance of 1.47 m for subject AV only. Each graph represents a single orientation of the reference bar, which is indicated in the top left corner. The solid line is a linear fit through the origin.

and the three repetitions appear as three points for each separation angle ζ. For $\zeta = \pm 30°$ and $\pm 15°$, six data points can be seen because there were two possible configurations of the stimuli for the same separation angle.

From figure 5 it can be seen that the dependence is approximately linear. Therefore the deviation $\Delta\phi$ is fitted to the equation $\Delta\phi = a\zeta$, where a is the slope. The squares of the correlation coefficients (R^2) are 0.01, 0.83, 0.69, 0.12, 0.77, and 0.82 in increasing order of the reference orientation. For the other subjects similar values are found. The reason for not including an offset is that it has no obvious physical meaning: an offset would mean that a test bar placed at the same position as the reference bar would not be set physically parallel. Therefore any offset we find has to be an artifact of the linear fit: if the results are in some way not linear, this would result in an offset. Moreover, including an offset in the fit does not affect the value of the slope a.

It can be seen in figure 5 that the slope of the fits is approximately zero for a reference orientation of 0° and 90°, whereas it is approximately 0.3 for the oblique reference orientations.

In figure 6 the slopes are plotted as a function of the reference orientation. Each graph corresponds to a different subject, marked in the top right corner. The different stimulus distances are indicated by diamonds for 1.47 m, stars for 2.10 m, and squares for 4.31 m.

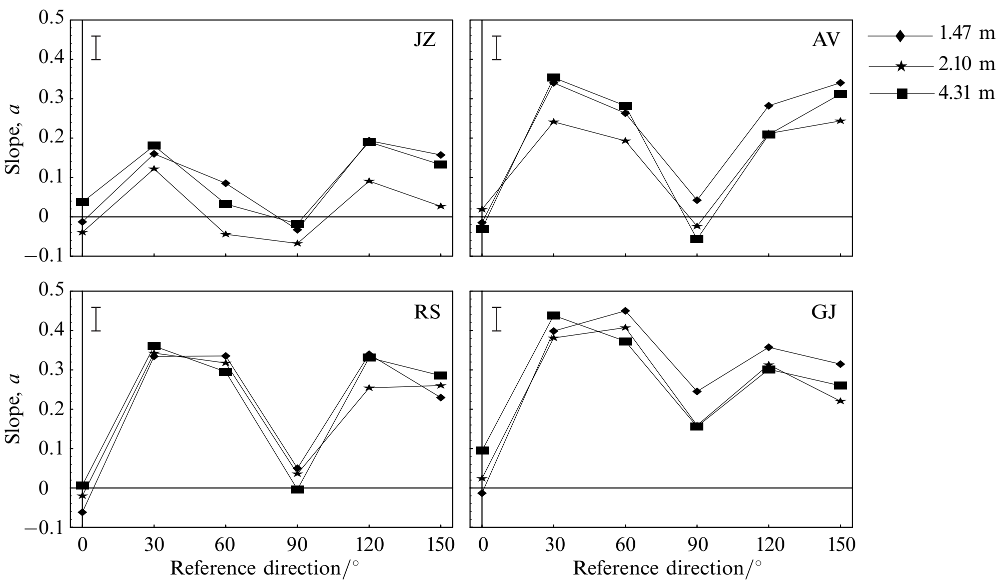

Figure 6. Slopes a of the fit $\Delta\phi = a\zeta$ for each subject (indicated in the top right corner) as a function of the reference orientation. The different stimulus distances are indicated by a line with diamonds for 1.47 m, with stars for 2.10 m, and with squares for 4.31 m. The error bar in the top left corner denotes the standard error.

Clearly the same pattern is present for all subjects and all distances: the slopes are large for all orientations of the reference bar, except for a 0° and 90° orientation where the deviations approach zero. For oblique reference orientations the slopes range from about 0.1 for subject JZ to 0.4 for subject GJ. This means that the orientations of the test and reference bars differ by 1° to 4° for every 10° separation. The deviation is counterclockwise when the test bar is positioned on the left of the reference bar and clockwise when it is on the right. Furthermore, the slopes are in good approximation independent of the stimulus distance. For subject JZ, there is a small departure from this pattern, because the slopes are nearly zero for a reference orientation of 60° as well, and the slope values are always lower for a distance of 2.10 m. However, the absolute slope values for JZ are the smallest of all four subjects. Therefore these data are more sensitive to noise.

The slopes and the significance levels for all distances and subjects are summarised in table 2. The slope values are significant for oblique reference orientations with a few exceptions for subject JZ, and are not significant for 0° and 90° reference orientations except for subject GJ where the slopes for a 90° angle are significant as well.

Table 2. Results of a linear fit of $\Delta\phi$ to $a\zeta$. The coefficient a is shown for each subject, distance, and reference orientation. The significance levels are indicated by a * for $p < 0.05$ and ** for $p < 0.01$.

$\phi_{\text{reference}}/°$	r/m	Subject JZ	AV	RS	GJ
0	1.47	−0.01	−0.01	−0.06*	−0.01
	2.10	−0.04*	0.02	−0.02	0.02
	4.31	0.04**	−0.03	0.01	0.09*
30	1.47	0.16**	0.34**	0.33**	0.40**
	2.10	0.12**	0.24**	0.34**	0.38**
	4.31	0.18**	0.35**	0.36**	0.44**
60	1.47	0.09*	0.26**	0.34**	0.45**
	2.10	−0.04	0.19**	0.32**	0.41**
	4.31	0.03	0.28**	0.30**	0.37**
90	1.47	−0.03	0.04*	0.05	0.25**
	2.10	−0.07**	−0.02	0.04	0.16**
	4.31	−0.02	−0.06**	−0.00	0.16**
120	1.47	0.19**	0.28**	0.34**	0.36**
	2.10	0.09**	0.21**	0.25**	0.31**
	4.31	0.19**	0.21**	0.33**	0.30**
150	1.47	0.16**	0.34**	0.23**	0.31**
	2.10	0.03	0.24**	0.26**	0.22**
	4.31	0.13**	0.31**	0.29**	0.26**

From table 2, it can be seen that the slopes are much smaller for the 0° and 90° orientations than for the other orientations for all subjects and all distances. Moreover, the slope is negligible in 8 out of 12 cases with a reference orientation of 0°, and 6 out of 12 for 90°. For oblique reference orientations, the slopes range from 0.1 to 0.4. These values are nearly constant for every subject. The vanishing slope for a reference orientation of 60° for subject JZ is an exception. Moreover, the slope values are in most cases independent of the distance for all subjects (see also figure 6): the standard error of the slope values is typically 0.03 and the corresponding 95%-confidence intervals are about 0.06 [$t_{0.025}(\text{df} = 35) = 2.03$].

4 Experiment 2

4.1 *Introduction*

A striking result of the previous experiment is that the deviations between the orientations of the test bar and the reference bar do not depend on the distance. At this point it is unclear what the implications are. But if we take, for example, a closed path with the corner points (1.47 m, 30°), (1.47 m, 0°), (4.31 m, 0°), and (4.31 m, 30°), we could find a measure for the mean curvature of visual space by asking a subject to subsequently set bars at those positions parallel to each other until the starting position is reached. The mean curvature is positive, zero, or negative if the angular defect between the last and the first bar is positive, zero, or negative. From experiment 1, we found that the deviations along the segments (1.47 m, 30°; 1.47 m, 0°) and (4.31 m, 0°; 4.31 m, 30°) cancel each other out. If we assume that there is no deviation from veridical along the median line, then there is no angular defect on three sides of the path. If visual space is curved, then the actual orientation of two bars at the endpoints of the remaining side which look parallel must be different. This can be tested experimentally by placing the reference bar and the test bar at varying distances from the observer along the remaining side of the closed path. Because the test bar and the reference bar would occlude each other if they were placed directly behind each other, a slightly different design is used: the reference bar is placed to the right of the observer ($\psi_{\text{reference}} = -30°$) and the test bar is placed to the left ($\psi_{\text{test}} = 30°$). If the results are independent of the distance of the bars from the subject, then this would mean that visual space is flat (zero curvature) or non-constantly curved with a zero mean.

4.2 *Method*

The experimental setup was almost identical to that used in experiment 1. In this case, the subjects entered the room not only blindfolded, but they were disoriented, having been rotated about their axis. Otherwise, the same procedure and task were applied. Different positions were used for the stimuli. The test bar was placed at varying distances (1.47 m, 2.10 m, 3.00 m, and 4.31 m) and at an angle of $\psi_{\text{test}} = 30°$ with respect to the observer. The reference bar was placed at either 1.47 m distance or 4.31 m at an angle $\psi_{\text{reference}} = -30°$ with respect to the observer (see figure 3). As in experiment 1, six different orientations of the reference bar were used (0°, 30°, 60°, 90°, 120°, and 150°) and each trial was repeated three times. Four different, naïve subjects participated. In total there were 144 trials for each subject. The experiment took about 9 h to complete.

The presentation order was as follows: after the positions of the test bar and the reference bar had been selected at random, the orientation of the test bar was measured for all six orientations of the reference bar in randomised order. This was repeated until all positions of the test bar and the reference bar had been measured once. The same procedure was applied three times with a different randomisation for each repetition. The order was also different for each subject.

4.3 *Results*

In figure 7, the results are shown graphically for subject AB, and are representative for the other subjects. Each diagram represents a top view of the experimentation room in which the orientation and position(s) of the test bars (thin lines) and reference bar (thick line) are indicated. The orientation of the test bar is the average of three measurements, and is indicated for all four distances from the subject (1.47 m, 2.10 m, 3.00 m, and 4.31 m). The distance of the reference bar from the subject is 1.47 m and 4.31 m in the left and right two columns respectively.

From figure 7 it can be seen that the deviations from veridical are always large. For this subject, these large deviations occur not only for oblique orientations of the reference bar, as was already demonstrated in experiment 1, but also for 0° and 90° (top row of figure 7). On the other hand, there are only minor differences between the orientations of the test bars for the different distances.

It is useful to plot the measured deviations $\Delta\phi$ as a function of the orientation of the reference bar, as is shown in figure 8. The rows show the results for the different subjects whose initials are indicated in the top right corner. In the left column the results are shown for a reference distance of 1.47 m and in the right column for a distance of 4.31 m. The different symbols refer to the different distances of the test bar from the observer: the diamonds correspond to a distance of 1.47 m, the stars to 2.10 m, the squares to 3.00 m, and the triangles to 4.31 m.

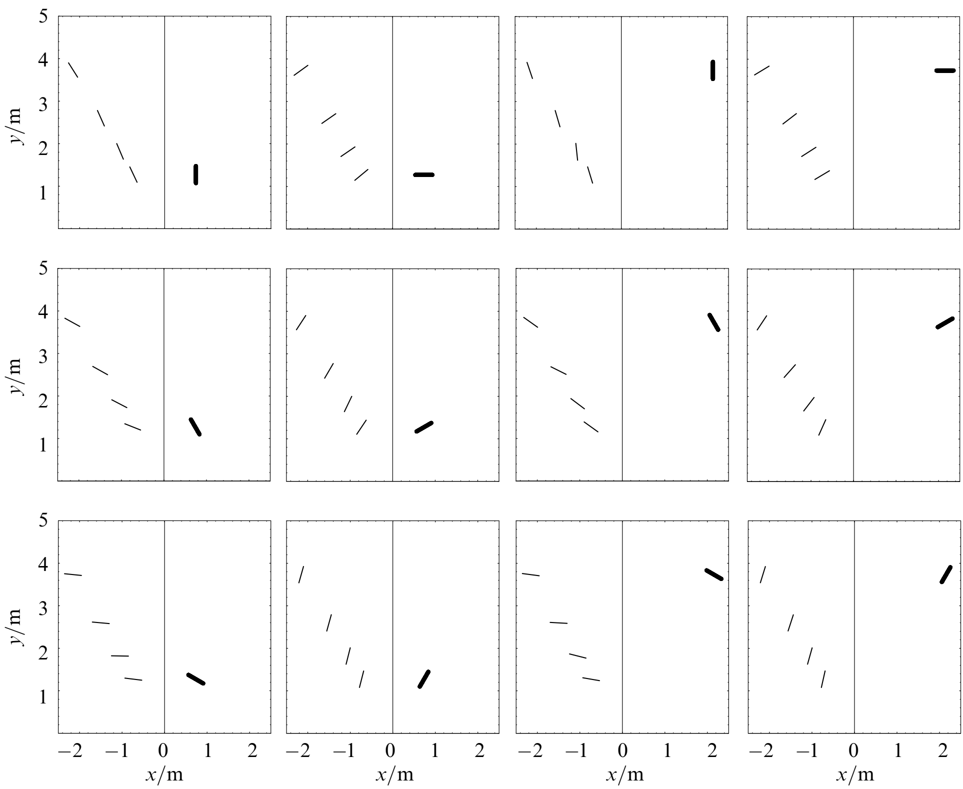

Figure 7. Results for subject AB in experiment 2. In each graph the orientations are shown for four positions of the test bar (thin lines) and for a given position and orientation of the reference bar (thick line). The subject is positioned at the origin. Each orientation of the test bar is the average of three measurements.

A comparison of the left and right columns reveals that the differences between the reference distances of 1.47 m and 4.31 m are negligible for subjects FS, AB, and EV. In addition, there is a considerable overlap between the data for the test bars at different distances, which means that the results are independent of the distance. In the case of subject SM (third row of figure 8), there are differences between the different distances of the test bar and the reference bar. However, the error bars are very large compared to those for the other subjects and there is no systematic pattern visible, which suggests that the differences are due to noise. We performed a multiway ANOVA to test the effect of the distance of each bar. There are no significant effects of the distances of the two bars ($F_{1,134} = 2.5039$, $p = 0.1141$ for the distance of the reference bar, and $F_{3,134} = 2.1964$, $p = 0.0875$ for the distance of the test bar).

The quantitative differences between the four subjects are considerable. For subject FS the deviations depend strongly on the reference orientation: the deviation vanishes for 0° and 90° and reaches a maximum deviation of about 35° for 30° and 120°. For subjects AB and EV the deviations are also large for 0° and 90° and nearly independent of the orientation. Both subjects have a smaller deviation for an orientation of 150° (about 15° and 30°, respectively) compared to the other orientations (about 30° and 40°, respectively). For this particular orientation, one end of the reference bar points directly between the eyes whereas the other end is invisible. The results for subject SM are different from those for the other subjects in that negative deviations occur as well. The negative deviations occur for all orientations of the reference bar except 30° and 120°.

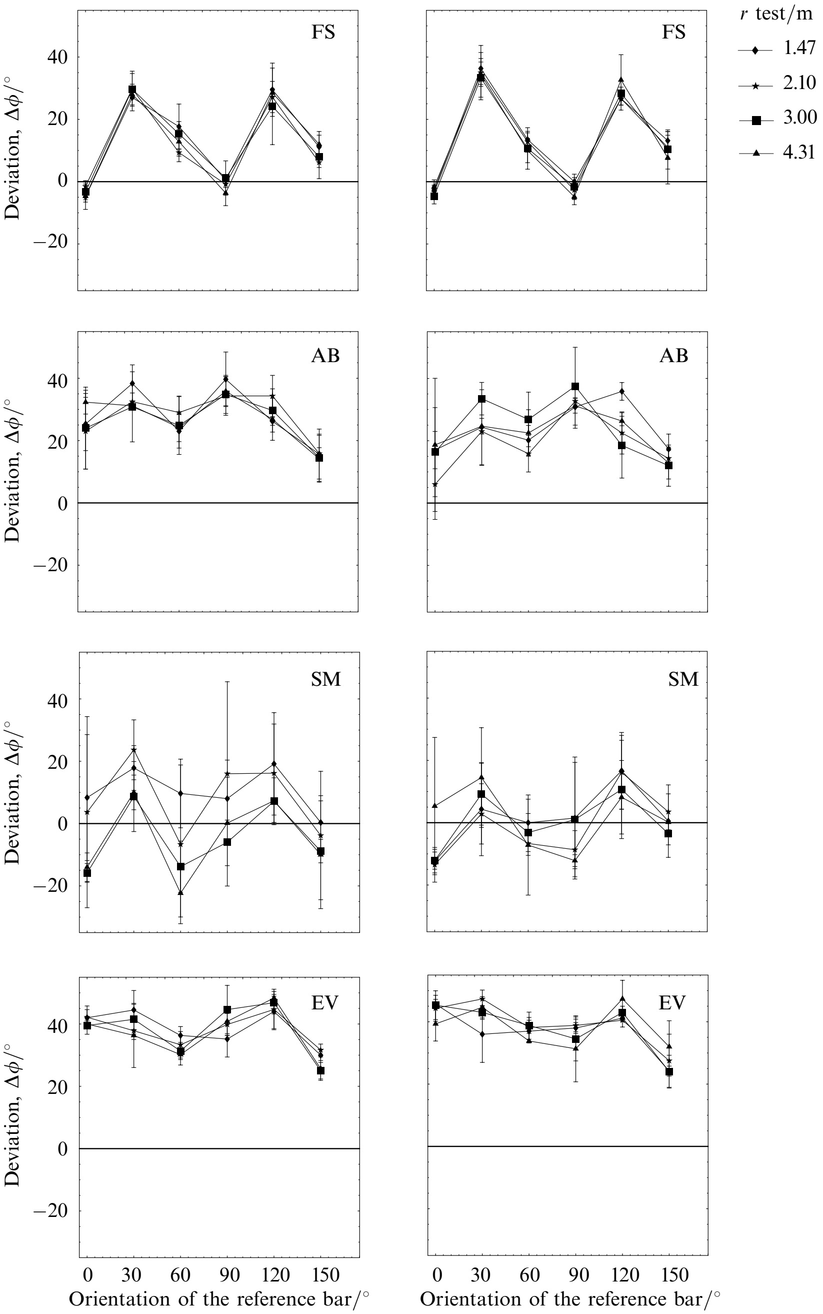

Figure 8. In each graph the measured deviation $\Delta\phi$ is shown as a function of the orientation of the reference bar for each distance of the test bar. The subject is indicated in the top right corner. The left and right columns depict the results for a distance of the reference bar from the subject of 1.47 m and 4.31 m, respectively.

5 General discussion and conclusions

From the results of experiment 1 it is clear that large systematic deviations of up to 40° occur between the orientation of the test bar and the reference bar. The size of the deviations is proportional to the separation between the test bar and the reference bar: the proportion ranges from 10% for subject JZ to 40% for subject GJ for the oblique orientations of the reference bar, and approaches zero for 0° and 90°, which corresponds to veridical settings. The magnitude of the deviations is not what one would expect on the basis of perspective: on that basis the deviation would be equal to the separation angle and, as a result, the slope values would be equal to 1. Instead, the slopes are 0.3 averaged over subjects, which is much closer to veridical. Nevertheless, the deviations are still large. The fact that there are large systematic deviations indicates that visual space is distorted in a systematic way, but it does not necessarily mean that visual space is a (constantly) curved space.

Similar results are found in experiment 2, where all subjects show large deviations. For subjects AB, SM, and EV these large deviations also occur for a reference orientation of 0° and 90°. The deviations measured for subjects AB and EV are nearly constant: only for an orientation of 150° has a smaller deviation been found. For this orientation one end of the reference bar is pointing directly between the eyes of the observer. Consequently, the other end of the reference bar is occluded by the disk. The subjects reported that in this situation the task was much harder, which may account for the different deviation. In the case of subject FS, the deviation is smaller (between 10° and 15°) for an orientation of 60° and 150° compared to a deviation of about 30° for an orientation of 30° and 120°. A possible explanation is that for these particular orientations a deviation of 30° would mean that the orientation of the test bar is 0° and 90° respectively. But these orientations already appear parallel to a reference orientation of 0° and 90°. So the deviations must be less if the results are to be consistent. The proportions can be estimated by dividing the deviation by the separation angle, which results in average values ranging from 20% for subject FS (for oblique orientations) to almost 70% for subject EV. For subject SM the average slope would be 0%.

The fact that for some subjects the large deviations occur for oblique orientations of the reference bar and not for 0° and 90° is reminiscent of the oblique effect reported in the literature (Appelle 1972). However, this oblique effect deals with orientations in a frontoparallel plane and not a horizontal plane at eye height as described here. To our knowledge the 'oblique effect' in the horizontal plane has not been reported before for a visual task. Surprisingly, very similar results have been obtained in haptic space: Kappers and Koenderink (1999) and Kappers (1999) report the same effect in a task where subjects need to set a test bar parallel to a reference bar haptically.

If, in a constantly curved Riemannian space, two vectors are parallel in the sense of Levi-Civita (Stoker 1969), then two vectors rotated over the same angle are also parallel. Thus, the deviation between the orientation of the test bar and the reference bar should be independent of the reference orientation. However, this is not the case. Therefore visual space cannot be described by a constantly curved Riemannian metric, and models such as Luneburg's cannot explain the data obtained from these experiments.

But how is it possible that a subject 'knows' when the bar orientation is parallel or perpendicular to the median line? This can be determined easily only if the reference bar is positioned on the median line. For then either the rod or the disk of the test bar is aligned with the line of sight. At other reference positions subjects will see the bar at some angle. Perhaps each subject has an internal reference, but in that case it is strange that this internal reference can only be used for two distinct orientations. Another explanation may be that the walls of the room or of the cabin provide an external reference (the fact that context has an influence on estimating orientations has already been demonstrated by Schoumans et al 2000). For instance, the side walls of the experimental room are parallel to the median line and the front wall is perpendicular to the median. A great deal of effort was taken to hide such external references by covering the cabin and the walls of the room with wrinkled plastic sheet material. Nonetheless, if subjects assumed that the experimental room was rectangular, which is very common in architecture, then it would still be possible to determine which orientation the room must have. The same reasoning holds for the sides of the cabin itself. Suppose that the environment provides sufficient external references so that for a 0° and a 90° reference orientation no deviation occurs (a zero slope). One would expect that if the external references were removed, the deviation between the test bar and reference bar would be independent of the reference orientation. As argued before, this is a necessary (though not sufficient) condition for describing visual space as a Riemannian space. Thus, in a context-free environment a Riemannian metric may still be a suitable description. This view is supported by the fact that in experiment 2 the deviation does not vanish for 0° and 90° for three subjects (AB, SM, EV), and that the deviations are indeed nearly independent of the reference orientation for two of these subjects (AB, EV). This suggests that this oblique effect is caused externally, because in experiment 2 subjects were disoriented before the measurement by being rotated blindfolded. Additionally, experiment 2 took only 2 h compared to 9 h in experiment 1, so subjects may not have had the time to recognise the layout of the room. As a consequence, a Riemannian metric may still be a valid description, but only in an unfamiliar and context-free room.

Another striking result is that the deviations between the orientations of the test bar and the reference bar do not depend on the distance from the subject. In order to investigate this further, experiment 2 was carried out. It was found that the deviations are independent of the distance of both the test bar and the reference bar from the observer. This has an important consequence for the curvature of visual space: if we consider a closed path with the corner points (1.47 m, 30°), (1.47 m, -30°), (4.31 m, -30°), and (4.31 m, 30°), we could find a measure of the mean curvature of visual space by asking a subject to subsequently set bars at those positions parallel to each other until the starting position is reached. From experiment 1 we find that the deviation is exactly opposite when going from (1.47 m, 30°) to (1.47 m, -30°) and when going from (4.31 m, -30°) to (4.31 m, 30°). From experiment 2 we find that there is no deviation in the radial directions from (4.31 m, 30°) to (1.47 m, 30°), and, assuming that visual space is symmetric about the median line, from (1.47 m, -30°) to (4.31 m, -30°). Thus, the angular defect between the last bar and the first bar is zero, meaning that the curvature is also zero and that visual space is intrinsically flat, or that the curvature is non-constant in the enclosed area with a zero mean, which seems unlikely.

A different interpretation is possible if we assume that the monocular retinal image is the only relevant cue for this task. This is supported by the fact that the change in orientation is much easier to detect monocularly (for a change in orientation of 5°, the average change of relative disparity ranges from 0.5 to 1.5 min of arc, whereas the monocular length change is on average 33 min of arc). In that case, only lateral displacement of the test and reference bars will have an effect on the measurements. Moreover, because the stimuli are scaled with the distance, the results will also be independent of the distance of the test and reference bars from the observer. As a consequence, the angular defect mentioned above will be zero because the proposed path encloses a zero area in visual space. Although this assumption automatically explains the results of both experiments, another question arises: how are subjects able to compensate for the laws of perspective? On the basis of perspective, one would expect the deviation to be equal to the separation angle because in this case the retinal projections are identical. However, a much smaller deviation is found (on average 30% of the separation angle). We do not know the answer to this question and we are not clear which interpretation is more suitable. Perhaps similar experiments in a monocular viewing condition in which the size of the bars is varied independently of the distance could provide some answers.

In summary, we come to the following conclusions: a Riemannian metric such as that proposed by Luneburg is not a suitable description in a familiar environment that contains external references. On the other hand, if it is assumed that the orientation dependence is caused by external references and that the dependence will vanish without them, Luneburg's model may nevertheless be an adequate description. In the classical experiments in a dark room almost all visual external references were removed.

The fact that the results are independent of the distance in both experiments indicates that visual space is either intrinsically flat in a context-free environment or non-constantly curved, or that the current task is essentially monocular. In the latter case, binocular disparity is the strongest cue for the distance, as in the classical experiments, but subjects ignore the distance when estimating the orientation of the bars.

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